Dynamics of opinion formation in a small-world network

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The dynamical process of opinion formation within a model using a local majority opinion updating rule is studied numerically in networks with the small-world geometrical property. The network is one in which shortcuts are added to randomly chosen pairs of nodes in an underlying regular lattice. The presence of a small number of shortcuts is found to shorten the time to reach a consensus significantly. The effects of having shortcuts in a lattice of fixed spatial dimension are shown to be analogous to that of increasing the spatial dimension in regular lattices. The shortening of the consensus time is shown to be related to the shortening of the mean shortest path as shortcuts are added. Results can also be translated into that of the dynamics of a spin system in a small-world network.

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I. INTRODUCTION

The physics of networks has received much attention in recent years. Topologically, a network consists of nodes and links, with the latter connecting the nodes in some fashion. Traditionally, the network models studied in many branches of science are the regular lattices and the random networks (classical random graphs) [1]. It was not until the late 1990s that scientists, many of them physicists, discovered that many real-world networks exhibit geometrical properties that are different from regular networks and random graphs [2–6]. Among these properties are: (i) A small average distance between arbitrarily chosen nodes in the network, i.e., each node may reach another node through a path that only passes through a few other nodes. This is the so-called small-world effect. Typically, the average distance increases with the number of nodes in the network only logarithmically. (ii) The clustering coefficient, which characterizes the extent in which the connected neighbors of a node are also connected, in real-world networks is relatively high. In the context of a society of individuals, these properties imply that one can approach any individual through a few intermediate connections, and the friends of a person are likely to be friends of each other.

An important branch of research in networks deals with the effects of the geometrical properties of networks on dynamical processes in networks. Taking the nodes as individuals, such dynamical processes may be epidemics in a population, cultural assimilation, opinion formation, voting or election, or decision making on competing for limited resources [7–16]. An up-to-date review on dynamical processes in complex networks can be found in the recent article by Bocaletti *et al.* [17]. An important question is to see how networking effects may affect the final state of a dynamical process and the time of reaching the final state. In the present work, we study the dynamical process of opinion formation based on a recent model of Redner and coworkers [9,10] in which the dynamics is based on a local majority rule [18]. Although the model was originally stated in terms of spins (nodes) that can take on one of two possible states, we will instead describe the model in terms of opinion formation between agents (nodes) who can take on one of two opinions. The translation from one description to another is obvious. The local majority rule then describes the influence of neighboring (connected) agents on an agent's opinion. In particular, we will study the changes in the dynamical process as a regular lattice is transformed into a small-world network by adding links to connect randomly chosen pairs of nodes [19]. This model of small-world network is analogous to that of Watts and Strogatz [20], and this underlying network has been used to investigate voting processes [11,12] and the Ising model [21]. We found that the time to reach a consensus drops sensitively with the addition of a small fraction of links to an otherwise regular lattice. The addition of links in a lattice of fixed spatial dimension is found to have similar effects as increasing the spatial dimensionality of a regular lattice and to bring the system closer to the mean field results previously obtained in the literature [9,10]. It should be noted that the model studied here is based on a local majority rule, which is different from another popular model of opinion formation called the voter model [11,13]. The latter has the advantage of being analytically manageable, while analytic treatments of the local majority model on complex networks remain a challenging task. While previous work on the voter model [13] focused on the effects of the spread in degrees in complex networks on the consensus time, here we found that there exists a strong correlation between the change in the time to reach a consensus and the mean shortest distance of the underlying network.

The plan of the paper is as follows. We introduce the model of opinion formation and the underlying network structure in Sec. II. Results of detailed simulations are presented and discussed in Sec. III. A summary of the results is given in Sec. IV.

II. OPINION FORMATION MODEL AND NETWORK STRUCTURE

We consider a system with N nodes which may, for example, represent N agents or N magnetic moments or spins depending on the situation under consideration. For each node, there are two possible states which are represented by

+1 and −1. These states represent two opposite opinions. Following the model studied by Redner and co-workers [9,10], the states of the nodes evolve in time according to the following updating rules. At each time step, one node is chosen randomly. The chosen node and his connected neighbors through an underlying network are then considered collectively for updating. For hypercubic lattices in D-dimension, for example, the cluster size is $(2D+1)$ nodes since each node is connected to $2D$ nearest neighboring nodes. All the nodes in the cluster of nodes will then be updated to take on the state of the local majority. The updating rule thus represents a consensus is reached in a cluster by taking the majority opinion. The procedure is then repeated until all the nodes reach a common state, i.e., when a final state of consensus is reached.

Previous studies [9,10] on the model were carried out on regular lattices. Recent research on the science of networks reveals that many real-world networks exhibit the small-world effect. The effect refers to the shortest distance from one node to another in networks. In contrast, the distance in a regular lattice grows with the size of the lattice. To study the effects of shortening of distances on opinion formation, we choose a lattice proposed by Newman and Watts [19] in which one may study the effects of a gradual change in the distance as links are randomly added to an underlying regular lattice. We start with a two-dimensional (2D) square lattice of size $\sqrt{N}\times\sqrt{N}$ with periodic boundary conditions. A number of additional links, called "shortcuts," are added between a randomly chosen pair of nodes. Slight modifications to the model in Ref. [19] are that the shortcuts cannot link a node to itself and at most one link is allowed between any two nodes, i.e., no doubly connected nodes. To quantify the number of additional links on the lattice, we introduce a parameter q, which is the number of additional links normalized by the total number of links $2N$ in the underlying lattice. For a given q, the number of additional links is $2qN$. Note that in changing a square lattice into a *fully connected* network without doubly connected nodes, a total of $N(N-5)/2$ additional links are needed. For a given q, the corresponding fraction of all possible additional links is, therefore, given by $4q/(N-5)$. From our numerical results, we found that additional links amounting to $q\approx 1$ are sufficient for studying the networking effects in the dynamics of opinion formation. Thus, we will focus in the range $0\leqslant q\lesssim 1$ in the following discussions. To allow for comparison with results on regular lattices, we randomly choose $2D$ neighbors among the neighbors of the chosen site for updating in each time step. We have checked that using the whole cluster of a chosen site for updating gives nearly identical results.

III. RESULTS

A. Shortening of consensus time

An important quantity for opinion formation is the consensus time, which is the time for the system to reach a common opinion. Starting with an initial configuration of a fraction p of +1 and a fraction of $(1-p)$ of −1 states among the nodes, the time to reach consensus is studied numerically.

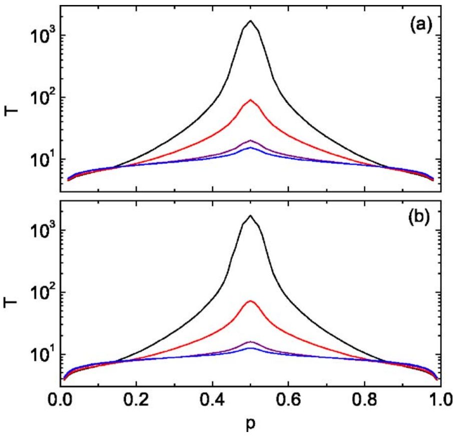

FIG. 1. (Color online) Consensus time T in units of Monte Carlo step per site as a function of the initial fraction p of +1 states in a $N=50\times 50$ square lattice for $q=0$, 0.1, 0.5, and 1, respectively (from top to bottom). (a) The updating group size is fixed to be $2D+1=5$ with the chosen node plus $2D$ randomly selected neighbors of the chosen node. (b) The updating process involves the chosen node and all its connected nodes.

Figure 1(a) gives the results of the mean consensus time T in a 2D lattice with different numbers of shortcuts in units of Monte Carlo step per site, i.e., one such time step corresponds to the time during which each node would have been updated once on average. Results are obtained by averaging 5000 runs with different initial configurations for given values of p and q. The $q=0$ results correspond to that on a square lattice. The results show that as a small number $(q<1)$ of shortcuts are added and the consensus time drops sensitivity for nearly the whole range of p. For a small range near $p\approx 0$ and $p\approx 1$, additional links may lead to a slightly longer consensus time, but just by a tiny bit, as a result of possible reinforcement of the survival of some scattered minority states by the additional links. In our model, we have chosen to fix the updating cluster size to be $(2D+1)$, where D is the dimension of the underlying regular lattice. It will be interesting to check the results against those in which the chosen node is updated together with *all* its connected nodes. Figure 1(b) shows the results for such a model, with the normalization to the simulation time steps taken to be $(\langle k\rangle+1)$, where $\langle k\rangle$ is the mean degree of the network after the shortcuts are added. The results of the two models are nearly identical. The reason is that the degree distribution of the network with added links, unlike the scale-free networks [22], has a sharp and characteristic peak. The drop in the consensus time is most sensitive in the intermediate range of p. For $p\approx 0.5$, a drop of consensus time of two orders of magnitude can be achieved by the addition of about N links to the system $(q=0.5)$ [see Fig. 1(a)]. Interestingly, recent studies on the dynamics of voter models [11,12] in one-dimensional (1D) small-world networks revealed that ordering processes become harder to achieve with the addition of shortcuts. The ordering processes in the Ising model was also found to be slower [21] in small-world networks. For the majority updating rule studied here, the shortcuts *accelerate* the ordering process of reaching a common opinion.

The sensitivity of the consensus time to q may be a result of a qualitative change in the time dependence of the dynam-

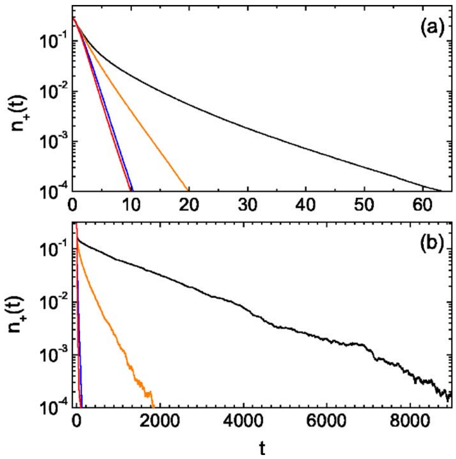

FIG. 2. (Color online) The fraction n_+ of nodes taking on the +1 state as a function of time t (in units of Monte Carlo step per site) for $p=0.3$ and $q=1$, 0.5, 0.1, and 0 (from left to right) in (a) a $N=50\times 50$ square lattice and (b) a $N=100$ 1D lattice.

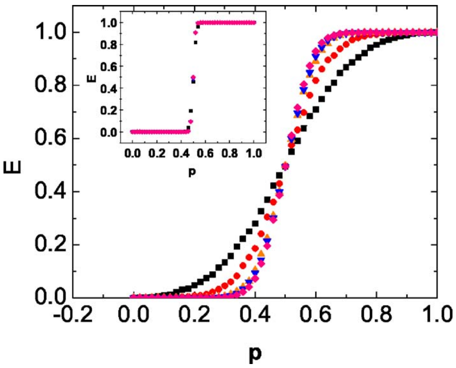

FIG. 3. (Color online) Exit probability E as a function of p for one-dimensional (main panel) and two-dimensional networks (inset) for different values of q. The 1D networks are of size $N=100$ nodes and the 2D networks are of size $N=50\times 50$. The symbols correspond to $q=0$ (squares), 0.1 (circles), 0.5 (triangles), and 1 (inverted triangles). The mean field results (diamonds) refer to those obtained by randomly choosing groups of $G=2D+1$ nodes for updating in each time step.

ics. We studied in detail how the number of nodes with a +1 state changes as a function of time for various values of p. Figure 2 shows the results for $p=0.3$ in both 2D and 1D networks, which are typical of the intermediate range of p. For $p=0.3$ in a 2D lattice with $N=2500$ nodes, the final consensus state has all the nodes with a −1 state. The fraction n_+ of nodes taking on the +1 state therefore drops as a function of time. For finite q, the drop in n_+ is much more rapid than $q=0$ and takes on an exponential decaying behavior. We have checked that similar q-dependence of $n_+(t)$ results in underlying lattice of higher spatial dimensions. It is useful to show the corresponding result in 1D [Fig. 2(b)], as results for $n_+(t)$ have also been reported in 1D networks for the voter model [11]. Comparing the results in 1D and 2D networks for the majority rule model, the behavior is qualitatively the same, but the time it takes for $n_+(t)$ to vanish is much longer in 1D. In general, the addition of shortcuts shortens the consensus time significantly. Comparing with results in the voter model [11], there are similarities as well as qualitative differences. For both models, the addition of shortcuts leads to a shorter consensus time when compared with a regular lattice ($q=0$). However, the two models are *qualitatively different* in the details of the dynamics. For the model based on local majority rule (see Fig. 2), $n_+(t)$ drops more rapidly with time for $q\neq 0$ for *all time t* during the time evolution, when compared with that in a regular lattice ($q=0$). For the voter model, there exists a significant duration of time (in particular the early stage) during the evolution of $n_+(t)$ that $n_+(t)$ drops *more slowly* in a network with shortcuts ($q\neq 0$) than in a regular lattice ($q=0$)—a result clearly shown in Fig. 3 of Ref. [11]. It is only at the later stage of the time evolution of $n_+(t)$ that $n_+(t)$ drops rapidly and the consensus time eventually becomes shorter than that in a regular lattice. While it is tempting to draw close analogies between the two models, it is also important to note that they are essentially different models and they show different dynamics in the process of evolving to a consensus.

B. Shortcuts and mean field limit

To further explore the effects of the shortcuts, we recall that in Ref. [9] a mean field limit of the opinion formation model was studied. The mean field limit corresponds to a model in which at each time step a group of G nodes are chosen *at random* among all the N nodes in the network for updating. The underlying network can thus be thought of as one in which every node is possibly connected to another. In Ref. [9], the authors compared simulation results in 1D, 2D, 3D, and 4D lattices without shortcuts and found that the behavior tends to approach the mean field limit as dimensionality increases. They found that the mean field behavior has not been reached in 4D and thus a higher upper critical dimension is expected for the problem. Figure 3 shows the exit probability, which is the probability that the final state of the system is one with all the nodes taking on the +1 state, as a function of p for 1D and 2D underlying lattices with added shortcuts. The inset of Fig. 3 shows the results of 2D networks. The $q=1$ results overlap with the mean field results, which are obtained by numerical simulations by randomly choosing groups of $G=5$ nodes for updating in every time step. Analytically, the exit probability takes on the form of an error function [9]. Our results, therefore, indicate that the effects of q in a lattice with *fixed* spatial dimension are similar to that of increasing dimensionality in regular lattices [19], i.e., increasing q has the effect of bringing the system closer to the mean field limit. To explore this effect more carefully, Fig. 3 (main panel) shows the results in a 1D network with 100 nodes for different values of q, together with results of the mean field limit. The updating group size is 3 for each time step. It is clear from the results that as q increases, the exit probability changes more abruptly near $p=0.5$ and the results approach the mean field results. The relation between an increasing q and an effective dimension has been studied by Newman and Watts [19]. It was pointed out that there exists a characteristic length scale ξ in the network model which gives the typical distance between the

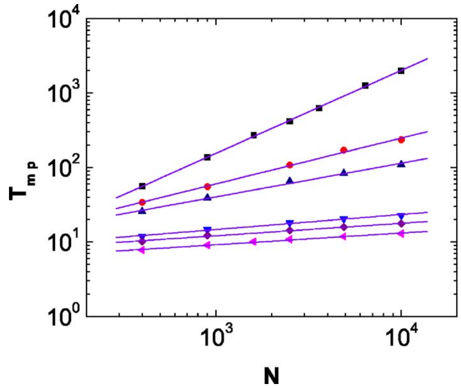

FIG. 4. (Color online) Most probable consensus time T_{mp} in units of Monte Carlo step per site as a function of network size N for the initial state corresponding to $p=0.5$. The data from top to bottom correspond to $q=0$, 0.05, 0.1, 0.5, 1, and the mean field limit, respectively. The lines represent linear fits with slopes 1.11 ± 0.01, 0.61 ± 0.02, 0.45 ± 0.02, 0.20 ± 0.01, 0.167 ± 0.007, and 0.158 ± 0.007.

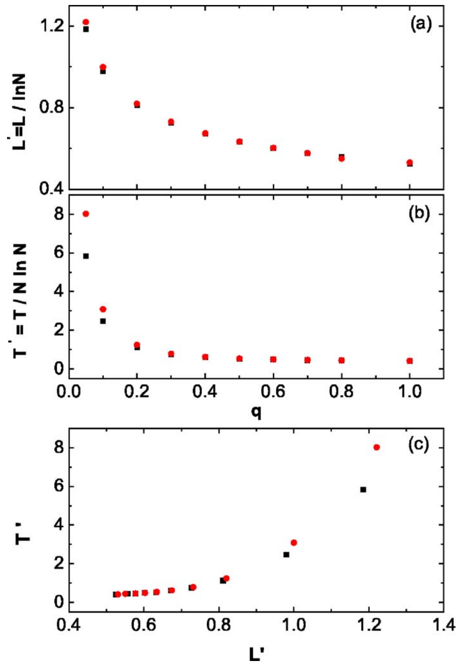

FIG. 5. (Color online) (a) The scaled mean shortest distance scaled $L'=L/(\ln N)$ as a function of q, for two-dimensional networks with $N=3600$ (squares) and $N=10\,000$ (circle) nodes. (b) The scaled mean consensus time $T'=T/(N\ln N)$ in units of Monte Carlo step per site as a function of q for initial state with $p=0.5$ for two-dimensional networks of sizes $N=3600$ (squares) and $N=10\,000$ (circles) nodes. (c) T' depends monotonically on L'.

ends of the shortcuts on the lattice. The length ξ plays the role of the correlation length in problems related to phase transitions and critical phenomena. The effective dimension can be extracted by imposing the relation $N\sim L^D$, where N is the number of nodes in the network and L is the mean shortest path between two nodes. For an underlying regular lattice in d-dimension of linear size S, the effective dimension D is related to d through [19] $D=d\ln(S/\xi)$ for $\xi\ll S$, and $D=d$ for $\xi\gg S$. Since $\xi\sim(qd)^{-1/d}$, $D=d$ only for a very dilute fraction of shortcuts, i.e., in the $q\to0$ limit. For a large range of q, we are in the regime of $\xi\ll S$ and hence the effective dimension D is higher than the dimension of the underlying lattice d.

The consensus time varies in different realizations corresponding to given values of q and p. The consensus time distribution for the $q=0$ (no added links) case consists of a mean peak and a long tail. The latter corresponds to runs in which clusters of minority states are formed so that a longer time is needed to change the minority opinion of the clusters. We define the *most probable* consensus time T_{mp} as the time corresponding to the peak of the distribution [9]. For different values of q we studied the distribution and obtained T_{mp} for networks of different sizes N. It is also observed that for q slightly larger than zero, the tail in the consensus time distribution shrinks rapidly, and hence the mean consensus time and T_{mp} become closer in value for finite q. Figure 4 shows the results of T_{mp} as a function of N for different values of q in a log-log plot, together with the results in the mean field limit [9]. The $q=0$ results reproduce the behavior reported in Ref. [9], with $T_{mp}\sim N^{\alpha}$ and $\alpha=1.11\pm0.01$. As q increases, the values of α decreases. The trend of decreasing α has been reported for regular lattices as dimensionality increases [9]. It was reported the mean field limit has not been reached in 4D regular lattices. In contrast, we found that even a small value of $q=1$ leads to a value of α quite close to the mean field limit. The observation is again related to the increasing effective dimension as q increases, as discussed above.

C. Consensus time and shortest distance

The mean field limit corresponds to networks with easy access from one node to another. The effects of adding links to a regular lattice lead to a similar effect in that the distances between nodes drop rapidly with the number of added links [19], thus leading to the small-world effect. This effect is illustrated in Fig. 5(a) by showing the drop in $L'=L/\ln N$ as a function of q for lattices of $N=3600$ and $N=10\,000$ nodes, where L is the mean shortest distance of the network obtained by averaging the shortest distances between all pairs of nodes in the network. Besides a $\ln N$ dependence, L drops with q. It is then interesting to correlate the mean consensus time and the shortest distance for a given value of q. In Fig. 5(b), the dependence of the mean consensus time $T'=T/(N\ln N)$ on q is shown for two values of N, for the case of $p=0.5$. Dividing the mean consensus time T by $N\ln N$ removes the N dependence and only q dependence remains. Note that both L' and T' show similar dependence on q. To illustrate that the shortening of the mean consensus time is intrinsically a result of the shortening of the shortest distance when shortcuts are added, we plot T' as a function of L' in Fig. 5(c). The data for small L' correspond to that of $q\gtrsim0.6$. We see that T' increases monotonically with L'.

IV. SUMMARY

We have studied the effects of adding shortcuts connecting randomly chosen pairs of sites in a regular lattice on the consensus time in reaching a common opinion within a model of local majority updating rule. The consensus time is found to drop sensitively with the addition of a small number of shortcuts. This drop is observed nearly over the whole range of p characterizing the initial distribution of opinions among the nodes. The rapid drop of consensus time with the addition of shortcuts is shown to be related to a qualitative change for finite q in the time dependence of the drop of the number of minority nodes as a function of time, as compared with regular lattice without shortcuts. We also compared our results with mean field results previously reported in the literature. It is found that the addition of shortcuts in a lattice of fixed spatial dimension has the similar effects of increasing the spatial dimension of regular lattices. This similarity is shown in both the behavior of the exit probability and the dependence of the most probable consensus time on the network size. Geometrically, the shortcuts decrease the mean shortest distance in a network and lead to the small-world effect. Dynamically, these shortcuts bring in the opinion of more connected nodes to a chosen node and hence have the effect of dissolving local minority groups more rapidly. The consensus time is found to change monotonically with the shortest distance in the network.

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